Commentary on [H 1914b]

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The mathematical content of this article is identical with what appears in HAUSDORFF's *Grundzüge* [H 1914a] on pages 401–402, 469–472. We note that the present article is dated 27. 02. 1914 and that the "Vorwort" of the *Grundzüge* carries the date 15. 03. 1914. The problem discussed was originally formulated by LEBESGUE in his 1902 thesis ([L 1902], p. 208); it consists of the following: To assign a non-negative real number $f(A)$ to each bounded subset A of $\mathbb{R}^n$ in such a way that:

(i) $f(E) = 1$ if E is the closed unit cube in $\mathbb{R}^n$;

(ii) $f(A) = f(B)$ if A, B are congruent;

(iii) $f(A \cup B) = f(A) + f(B)$ if A, B are disjoint;

(iv) $f(A_1 \cup A_2 \cup \cdots) = f(A_1) + f(A_2) + \cdots$ if $A_1, A_2, \ldots$ is any denumerable sequence of mutually disjoint sets whose union is bounded.

The congruence condition in (ii) is to be interpreted as follows: A, B are congruent if there exists $\rho \in G_n$, the Euclidean group of (Euclidean) distance preserving transformations in $\mathbb{R}^n$, such that $\rho(A) = B$. Let us call the problem of the existence of such an f, the *σ-additive measure problem in $\mathbb{R}^n$* and the problem of the existence of an f verifying only the conditions (i), (ii), (iii), the *finitely additive measure problem in $\mathbb{R}^n$*. The two major results of HAUSDORFF in this paper can then be formulated as follows: (a) *the σ-additive measure problem in $\mathbb{R}^n$ has no solution for any $n \geq 1$;* (b) *the finitely additive measure problem in $\mathbb{R}^n$ has no solution if $n \geq 3$.* Both of these results are proven by using the axiom of choice; contrary to many of his contemporaries HAUSDORFF always remained an unrepentant user of this axiom. At the end of the paper, HAUSDORFF mentions explicitly that the finitely additive measure problem in $\mathbb{R}^1$ and $\mathbb{R}^2$ remains open since his analysis does not apply to the groups G_1, G_2, the Euclidean groups in $\mathbb{R}^1, \mathbb{R}^2$ respectively. Let us state right away that BANACH showed, in 1923, that the finitely additive measure problem in $\mathbb{R}^1, \mathbb{R}^2$, does have infinitely many solutions ([B 1923]).

Let us recall that LEBESGUE had left the σ-additive measure problem in $\mathbb{R}^n$ unresolved; his construction had proved the existence of $f(A)$ for the so-called LEBESGUE-measurable bounded subsets of $\mathbb{R}^n$ and he had left the existence of non-measurable subsets as an open question. As mentioned by HAUSDORFF, this was first settled by VITALI (in 1905) by the construction of a non-measurable set in $\mathbb{R}^1$. A point of historical interest is the second footnote on p. 428; after having stated in the text that the first such example is due to VITALI, HAUSDORFF indicates in the footnote, among other references, that the example in the text is due to himself. It is clear that HAUSDORFF had discovered it independently of VITALI since the latter's 1905 paper on the subject [V 1905] appears to have been seen by very few contemporaries; VITALI's short paper (the actual text, in Italian, is only 2 1/2 pages long) seems to have been printed privately and was not to be found in any regular mathematical journal. Be that as it may, HAUSDORFF's method is essentially the same as that of VITALI; HAUSDORFF uses the subgroup $G_\delta = \{n\delta, n \in \mathbb{Z}\}$, δ being a fixed irrational number, whereas VITALI works with the subgroup of rational numbers. This method of using a denumerable dense subgroup of the additive group $\mathbb{R}$ (in HAUSDORFF's case, the dense subgroup is $G = G_\delta + \mathbb{Z}$) is probably the most widely disseminated one in the existing text-books, old and new, for the construction of non-measurable sets in $\mathbb{R}$. HAUSDORFF concludes that the σ-additive measure problem in $\mathbb{R}^1$ (and hence in $\mathbb{R}^n, n \geq 1$) has no solution.

The proof of the impossibility of the finitely additive measure problem in $\mathbb{R}^n, n \geq 3$ (statement (b) above) is what has made this paper a mathematical landmark. As in the case of (a), HAUSDORFF reduces the problem rather summarily (but correctly, see below) to one on the unit sphere; for $K = K_2 = S^2 \subset \mathbb{R}^3$, he then produces the so-called *Hausdorff paradoxical decomposition*:

$$K = A \cup B \cup C \cup Q \tag{1}$$

where A, B, C, Q are four disjoint subsets of K, Q being denumerable and

$$A \sim B \sim C \sim B \cup C \tag{2}$$

the congruence $\sim$ here being under the group of rotations SO_3. A decomposition (1) excludes the possibility of having an SO_3- invariant finitely additive positive set function f defined for all subsets of K with $f(K) > 0$; indeed for such an $f, f(Q)$ must be 0 (an easy argument given on page 432) and

$$f(A) = f(B) = f(C) = f(B \cup C) = f(B) + f(C)$$

whence all of these numbers are 0 which is impossible since

$$0 < f(K) = f(A) + f(B) + f(C).$$

The decomposition (1) is obtained by the consideration of a denumerable subgroup $G = G(\varphi, \psi)$ of SO_3 generated by two rotations φ, ψ such that

$$\varphi^2 = 1,\ \psi^3 = 1,$$

1 being the identity map, and such that φ, ψ *satisfy no other non-trivial relations*. Such a group G can then be written as a *disjoint union* of three subsets,

$$G = \tilde{A} \cup \tilde{B} \cup \tilde{C} \tag{3}$$

in such a way that

$$\varphi(\tilde{A}) = \tilde{B} \cup \tilde{C}, \;\; \psi(\tilde{A}) = \tilde{B}, \;\; \psi(\tilde{B}) = \tilde{C}, \;\; \psi(\tilde{C}) = \tilde{A};$$

this is done fairly simply on pages 432–433 (except that we write $\varphi\rho$ for HAUSDORFF's $\rho\varphi$ so that for us $\varphi\rho(x) = \varphi(\rho(x))$ etc.; also $\varphi(\tilde{A}) = \{\varphi\rho : \rho \in \tilde{A}\}$ etc.). From (3) the decomposition (1) is easy to obtain; take Q to be the denumerable set formed of all the fixed points in K of all the rotations $\rho \neq 1$ in G (two such fixed points for each ρ) and write $P = K \setminus Q$; let M be a subset of P obtained by choosing one point from each orbit

$$\{\rho(x) : \rho \in G\}, \;\; x \in P \subset K;$$

now write $A = \tilde{A}M, \;\; B = \tilde{B}M, \;\; C = \tilde{C}M$ where, for any subset $S \subset G$,

$$SM = \{\rho(x) : \rho \in S, \;\; x \in M\}.$$

It is then easy to see that A, B, C, Q provide a decomposition (1) with

$$\varphi(A) = B \cup C, \quad \psi(A) = B, \quad \psi(B) = C$$

so that (2) is verified.

A very detailed and readable account of HAUSDORFF's proof for the existence of his paradoxical decomposition (1) is given by SIERPIŃSKI ([S 1954]) where a small gap in HAUSDORFF's construction of the group $G(\varphi, \psi)$ is filled; [S 1954] contains an elementary exposition of other paradoxical decompositions as well, one of which we shall mention in the following.

The proof of the existence of $G = G(\varphi, \psi)$ is an important element of HAUSDORFF's paper; later authors have given simpler constructions (see [W 1993] p. 20 for references). Indeed, it is possible to give explicitly two rotations φ_1, φ_2 which generate a free subgroup F_2 with two generators in SO_3 ([W 1993], p. 15); using such an F_2 in place of $G = G(\varphi, \psi)$ above, the decomposition (3) is even easier to obtain and then the whole proof can be worked out exactly as before. However, it should be pointed out that a group like $G(\varphi, \psi)$ (a free product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$) must necessarily contain a free subgroup with two generators, a fact noticed by several authors (in particular by VON NEUMANN in his important 1929 paper [vN 1929], p. 606). HAUSDORFF's own proof for the existence of $G(\varphi, \psi)$ consists of choosing ψ as a $2\pi/3$-rotation around the z-axis and φ as a π-rotation around an axis (through the origin) in the xz-plane making an angle of $\theta/2$ with the z-axis where θ is chosen suitably (x, y, z here denoting the traditional perpendicular axes in $\mathbb{R}^3$). It is enough to have θ such that $\cos\theta$ is transcendental, although HAUSDORFF does not say this; he shows directly that by avoiding an at most denumerable set of angles, a suitable θ can be determined. HAUSDORFF seems to consider it much too obvious and hence not worth mentioning explicitly that the non-existence of an SO_3-invariant finitely additive non-trivial positive measure defined for all subsets of $K = S^2$ implies the impossibility of the finitely additive measure problem in $\mathbb{R}^3$. This implication can be seen as follows: if there were a finitely additive measure f verifying (i), (ii), (iii) in $\mathbb{R}^3$ then, for $A \subset K$,

$$g(A) = f(\{tx : x \in A,\ 0 < t \leq 1\})$$

would define a non-trivial finitely additive positive SO_3-invariant measure g defined for all subsets of K which would be a contradiction. HAUSDORFF does point out the easy argument which shows that if the finitely additive measure problem is impossible in $\mathbb{R}^3$ then it is also impossible in all $\mathbb{R}^n, n \geq 3$.

The direct and indirect effects of this short paper of HAUSDORFF have been remarkable. As mentioned above, it prodded BANACH to solve (in 1923) the finitely additive measure problem in $\mathbb{R}^1$ and in $\mathbb{R}^2$; here BANACH used methods which were to be fruitfully generalized by VON NEUMANN and others later (in the guise of *amenable* groups). On the other hand, soon afterwards in 1924, BANACH and TARSKI [BT 1924] gave a very striking theorem in $\mathbb{R}^n$, $n \geq 3$, which establishes the existence of even more surprising decompositions; their theorem gives that if *A, B are any two bounded subsets of $\mathbb{R}^n, n \geq 3$, with non-empty interiors, then, they are equivalent under finite decomposition using the group G_n* i. e. for some integer $m \geq 1$,

$$A = A_1 \cup \cdots \cup A_m,\ B = B_1 \cup \cdots \cup B_m,\ A_i \sim B_i,\ i = 1, \ldots, m \qquad (4)$$

where $A_1, \ldots, A_m$ is a disjoint partition of A, $B_1, \ldots, B_m$ is a disjoint partition of B and the congruence $A_i \sim B_i$ is with respect to the Euclidean group G_n; they also prove that such a theorem is false for $n = 1, 2$. BANACH and TARSKI give corresponding results for $S^n, n \geq 2$, and S^1 using the congruence group O_n (or SO_n). Decompositions like (4) form what is generally referred to as the BANACH-TARSKI paradox. BANACH and TARSKI based their proof on HAUSDORFF's paradoxical decomposition (1) of S^2 and used a general set-theoretical theorem due to BANACH [B 1924], sometimes called the BANACH-SCHRÖDER-BERNSTEIN theorem since it generalizes the usual SCHRÖDER-BERNSTEIN theorem of set theory.

As regards the finitely additive measure problem the first general analysis valid in abstract spaces was provided by VON NEUMANN in 1929 [vN 1929]; he considered an abstract set X on which acted a group G and studied the problem of the existence of a finitely additive measure $\mu(A)$, with $0 \leq \mu(A) \leq \infty$, defined for every subset A of X which has the property of being G-invariant ($\mu(A) = \mu(\varphi(A)), \varphi \in G, A \subset X$) and which is such that for some fixed non-empty set $E \subset X$, one has $\mu(E) = 1$. Let us call (with VON NEUMANN) such a μ, if it exists, a $[X, E, G]$-measure; one of the merits of VON NEUMANN's analysis is to reduce the problem of the existence of a $[X, E, G]$-measure to a purely group-theoretical problem of the existence of a $[G, G, G]$-measure where G acts on itself say by left multiplication. A group G for which a $[G, G, G]$-measure exists is called (in modern terminology) an *amenable* (more exactly, left amenable) group; VON NEUMANN himself called such groups "messbar".

von Neumann then proves that if G is amenable and a certain general condition is satisfied by X and E (a condition easily verified if, $X = \mathbb{R}^n$, $E =$ closed unit cube in $\mathbb{R}^n$, $G = G_n$ and other similar classical examples) then there is a $[X, E, G]$-measure. In this proof, von Neumann uses Banach's method as in [B 1923]. If on the other hand G has a free subgroup with two generators F_2 (and then G is not amenable) then by following the Hausdorff-Banach-Tarski procedure, von Neumann shows (under a very general condition on E) that no $[X, E, G]$-measure can exist. Von Neumann then deduces a number of classical existence and non-existence results from his general theory by proving first that the Euclidean groups G_1, G_2 are amenable whereas $G_n, n \geq 3$, contains a free subgroup F_2. Von Neumann's analysis was soon completed by a very elegant general theorem of Tarski (first announced in 1929) which can be stated as follows: *a $[X, E, G]$-measure exists if and only if E is not G-paradoxical* [T 1929]. We say that $E \subset X$ is *G-paradoxical* if for some integers m, n there are mutually disjoint subsets $A_1, \ldots, A_m, B_1, \ldots, B_n$ in E and $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n$ in G such that

$$E = \bigcup_{i=1}^{m} \varphi_i(A_i) = \bigcup_{j=1}^{n} \psi_j(B_j). \tag{5}$$

It is easy to see that if E is G-paradoxical then a $[X, E, G]$-measure μ cannot exist; indeed if (5) holds, such a μ must satisfy

$$\mu(E) \leq \sum_{i=1}^{m} \mu(\varphi_i(A_i)) = \sum_{i=1}^{m} \mu(A_i) = \mu\left(\bigcup_{i=1}^{m} A_i\right)$$

$$\mu(E) \leq \sum_{j=1}^{n} \mu(\psi_j(B_j)) = \sum_{j=1}^{n} \mu(B_j) = \mu\left(\bigcup_{j=1}^{n} B_j\right);$$

but by the disjointness of all the A_i's and the B_j's in E we have

$$\mu(E) \geq \mu\left(\bigcup_{i=1}^{m} A_i\right) + \mu\left(\bigcup_{j=1}^{n} B_j\right) \geq 2\mu(E)$$

which is impossible since $\mu(E)$ must be 1. The proof of the converse (i. e. if E is not G-paradoxical then a $[X, E, G]$-measure μ exists) is very much more subtle; Tarski gave a proof of a more general result in 1938 in [T 1938] by a careful refinement of his own work with Banach which in turn was so crucially influenced by Hausdorff's. The study of the finitely additive measure problem in $\mathbb{R}^n$ can be completely accomplished via Tarski's general theorem. A readable modern exposition of Tarski's theorem and numerous related problems (many unsolved) can be found in [W 1993].

Thus we see that this short paper of Hausdorff has given rise to the subject of amenable groups and to that of paradoxical decompositions. Each subject has developed a huge literature; we indicate briefly some recent publications which could serve as further guide to the current developments. A recommended monograph is that of WAGON [W 1993] which contains a very readable exposition of both subjects and a substantial bibliography (upto 1993).

As VON NEUMANN had already noted, the amenability of a group G is equivalent to the existence of a left G-invariant positive linear functional M on the Banach space $B(G)$ of all real-valued bounded functions on G with $M(1) = 1$. If G is a general topological group (the preceding case being that of G with the discrete topology) then one could ask for the existence of a left (or right or two-sided) G-invariant positive linear functional M on other G-invariant topological vector spaces of functions (or other objects like measures or distributions) defined on G. This theory has far-reaching implications (specially for G locally compact or G a LIE group) on the representation theory of G. It is then an important matter to settle the relationship between the different types of amenability which thus arise. A relatively recent account of some of this is given in the monograph of PATERSON [P 1988] which contains references to other relevant work on amenability. One of the most important by-products emanating from HAUSDORFF's paradoxical decomposition, via VON NEUMANN's analysis, is the notion of amenability which seems to play an ever-increasing role in diverse areas of mathematics. To mention one important area of application, let us point out the theory of operator algebras which forms the basis of A. CONNES' ambitious programme of non-commutative geometry; see, for example, his book [Co 1994], section V. 7.

Much work has been devoted to determining how "regular" the sets in the HAUSDORFF decomposition (1) or the BANACH-TARSKI decomposition (4) can be. A very important recent progress is contained in the paper of DOUGHERTY and FORMAN [DF 1994]; they prove (as a special case of a much more general theorem) that the sets in the decompositions (1) and (4) can be chosen to have the property of BAIRE (provided, in (4), A, B have this property). This solves a problem concerning the existence of MARCZEWSKI measures (explained in [W 1993]). LACZKOVICH [La 1992] gives a report on several other interesting and novel aspects of paradoxical decompositions.

A recent article which surveys (and gives complete proofs) of several themes concerning amenability and paradoxical decompositions is [CGH 1999].

HAUSDORFF himself never published anything more on the subject; however, from his Nachlass we know that he followed with interest the work of BANACH and TARSKI. In [NL HAUSDORFF: Fasz. 1028] (dated 19. 6. 1924, 25. 6. 1924) HAUSDORFF gives a short proof of the BANACH-TARSKI paradox in $\mathbb{R}^3$ by using his own paradoxical decomposition of S^2, a simple form of the abovementioned BANACH-SCHRÖDER-BERNSTEIN theorem (HAUSDORFF calls this an analogue of the BERNSTEIN equivalence theorem) and the fact (which he proves) that $S^2 \equiv S^2 \setminus D$ where D is any denumerable set and $\equiv$ stands for equivalence under finite decomposition (using the group SO_3). A similar proof was published later by SIERPIŃSKI and it appears in [S 1954], p. 93. A novel feature of this manuscript is that HAUSDORFF obtains his paradoxical decomposition by using a subgroup $G(\alpha,\beta,\gamma)$ of SO_3 where $\alpha^2=\beta^2=\gamma^2=1$, α,β,γ being otherwise independent; thus $G(\alpha,\beta,\gamma)$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, each α,β,γ being a suitable π-rotation. The paradoxical decomposition is obtained by exactly the same method as the one outlined above except that now one has 5 disjoint sets, one of them denumerable as before. One reason why HAUSDORFF prefers this method of arriving at his decomposition seems to lie in the simple and elegant algebra of 3×3 matrices underlying the choice of the π-rotations α,β,γ. Indeed, he returns to this matrix algebra in greater detail in [NL HAUSDORFF: Fasz. 386], dated 19. 2. 30. I must admit that the matrix algebra concerned is very pleasant to work out. However, given that the only point relevant to the existence of his paradoxical decomposition is the existence of suitable free products in SO_3 for which there are several shorter proofs available now (see references given above), I refrain from reproducing any parts of these manuscripts.

References

[B 1923] BANACH, S.: *Sur le problème de la mesure.* Fund. Math. 4 (1923), 7–33, *Oeuvres*, vol. 1, PWN, Warszawa 1967, 66–89.

[B 1924] BANACH, S.: *Un théorème sur les transformations biunivoques.* Fund. Math. 6 (1924), 236–239. *Oeuvres*, vol. 1, PWN, Warszawa 1967, 114–117.

[BT 1924] BANACH, S.; TARSKI, A.: *Sur la décomposition des ensembles de points en parties respectivement congruentes.* Fund. Math. 6 (1924), 244–277. *Oeuvres,* vol. 1, PWN, Warszawa 1967, 117–148.

[Co 1994] CONNES, A.: *Noncommutative geometry.* Academic Press, New York 1994.

[GCH 1999] CECCHERINI-SILBERSTEIN, T.; GRIGORCHUK, R.; DE LA HARPE, P.: *Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces.* Proceedings of the Steklov Institute of Mathematics 224 (1999), 57–97.

[DF 1994] DOUGHERTY, R.; FOREMAN, M.: *Banach-Tarski decompositions using sets with the property of Baire.* J. Amer. Math. Soc. 7 (1994), 75–124.

[L 1902] LEBESGUE, H.: *Intégrale, longueur, aire.* Annali di Math. pura ed appl. (3) 7 (1902), 231–259. *Oeuvres Sc.,* vol. 1, Ens. Math., Genève 1972, 203–331.

[La 1992] LACZKOVICH, M.: *Paradoxical decompositions : a survey of recent results.* Proceedings of the First European Congress of Mathematics, Paris 1992, vol. 2, 159–184.

[vN 1929] von Neumann, J.: *Zur allgemeinen Theorie des Masses.* Fund. Math. 13 (1929), 73–116. *Collected Works*, vol. 1, Pergamon, Oxford 1961, 599–642.

[P 1988] Paterson, A. T.: *Amenability.* American Mathematical Society, Providence, Rhode Island 1988.

[S 1954] Sierpiński, W.: *On the congruence of sets and their equivalence by finite decomposition.* Lucknow Univ., Lucknow 1954. Reprinted in Congruence of sets and other monographs, Chelsea, New York 1967.

[T 1929] Tarski, A.: *Sur les fonctions additives dans les classes abstraites et leurs applications au problème de la mesure.* C. R. Séances Soc. Sci. Lettres Varsovie, Cl. III 22 (1929), 114–117. *Collected Papers*, vol. 1, Birkhäuser, Basel 1986, 245–248.

[T 1938] Tarski, A.: *Algebraische Fassung des Massproblems.* Fund. Math. 31 (1938), 47–66. *Collected Papers*, vol. 2, Birkäuser, Basel 1986, 453–472.

[V 1905] Vitali, G.: *Sul problema della misura dei gruppi di punti di una retta.* Tip. Gamberini e Parmeggiani, Bologna 1905. *Opere*, Ed. Cremonese, Bologna 1984, 231–235.

[W 1993] Wagon, S.: *The Banach-Tarski paradox.* Cambridge University Press, Cambridge 1985. First paperback edition (corrected reprint) 1993.