Commentary on [H 1919a]

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Since the explosion of the theory of fractals in the 1970's, this [H 1919a] is probably the most often cited paper of Hausdorff, at least in the popular and semi-popular scientific literature; it is therefore not surprising that its contents are often misquoted and misinterpreted. Hence, we summarize first the mathematical novelties contained in this paper; we shall do so in the language of general metric spaces (X, ρ), although Hausdorff restricted himself to $X = \mathbb{R}^q$ equipped with its usual q-dimensional Euclidean distance, $q = 1, 2, \ldots$

Let $\mathcal{U}$ be any family of bounded subsets of the metric space (X, ρ) such that for any $\varepsilon > 0$, every subset A of X can be covered by a finite or denumerable family of sets U in $\mathcal{U}$ with $d(U) < \varepsilon$, $d(U)$ being the diameter of the set U; this last condition on the family $\mathcal{U}$ can be omitted but its absence may lead to trivialities. Suppose that to each U in $\mathcal{U}$ is assigned a non-negative real number $\ell(U)$; define, for $A \subset X, \varepsilon > 0$,

$$L_\varepsilon(A) = inf \left\{ \sum_{n \geq 1} \ell(U_n) : A \subset \bigcup_{n \geq 1} U_n, \ \ U_n \in \mathcal{U}, \ \ d(U_n) < \varepsilon \right\} \tag{1}$$

and

$$L(A) = \lim_{\varepsilon \downarrow 0} L_\varepsilon(A). \tag{2}$$

To avoid uninteresting difficulties it is best to assume that $\emptyset \in \mathcal{U}$ and $\ell(\emptyset) = 0$, although Hausdorff does not do so explicitly; thus $L_\varepsilon(\emptyset) = 0 = L(\emptyset)$; notice that $0 \leq L_\varepsilon(A) \leq \infty$ and that, for obvious reasons, $L_\varepsilon(A)$ increases with decreasing ε i. e. if $0 < \varepsilon_1 < \varepsilon_2$ then

$$L_{\varepsilon_1}(A) \geq L_{\varepsilon_2}(A).$$

Hausdorff's first observation is that $A \mapsto L(A)$ defines a *metric outer measure* on the family of all subsets A of the metric space (X, ρ); explicitly, this means that L verifies the conditions I–IV given below:

(I) $0 \leq L(A) \leq \infty$ (with $L(\emptyset) = 0$)

(II) If $B \subset A$ then $L(B) \leq L(A)$.

(III) If $A = A_1 \cup A_2 \cup \cdots$ then $L(A) \leq \sum_{n \geq 1} L(A_n)$.

(IV) If $\delta(A, B) > 0$ ($\delta(A, B)$ being the distance between the non-empty subsets A, B) then

$$L(A \cup B) = L(A) + L(B).$$

Further if the sets in $\mathcal{U}$ are Borel sets then L satisfies the following as well:

(V) $L(A) = \inf\{L(B) : B \supset A, \quad B \text{ being } L\text{-measurable}\}$

where B is called L-measurable if for any subset W of X,

$$L(W) = L(W \cap B) + L(W \setminus B). \tag{3}$$

Recall that the Borel sets of X are defined to be the sets belonging to the smallest σ-algebra generated by the open sets of X. According to current terminology, any set function L, defined for all subsets of a set X, satisfying I, II and III, is called an *outer measure* in X; if X is a metric space and the outer measure L in X satisfies IV then L is called a *metric outer measure* in X; finally, any outer measure L in any set X which satisfies V is called a *regular outer measure* in X. The condition (3) is called the Carathéodory condition of measurability and was first introduced in [C 1914]. Although Carathéodory (like Hausdorff) supposed that $X = \mathbb{R}^q$, it was obvious that all his work involving only I, II, III was valid for any outer measure in any set X and that all his considerations involving IV applied to any metric space X; for Carathéodory, the purpose of condition IV was to ensure that at least all the Borel sets of the metric space X were measurable. As observed by Hausdorff (p. 158), if L satisfies I–IV then a simple modification of L provides other metric outer measures M, N which satisfy V as well. In [C 1914] Carathéodory showed that if $\mathcal{U}$ is the class of all bounded subsets of $X = \mathbb{R}^1$ and $\ell(U) = d(U)$, the diameter of U, then the corresponding set function $L(A)$ (called the *linear measure* of A) constructed via (1) and (2) verifies the conditions I–V; as Hausdorff remarks on p. 159, Carathéodory's proof applies verbatim to the general situation envisaged by Hausdorff. In [C 1914] Carathéodory further points out that for the case of the linear measure the sets in the family $\mathcal{U}$ can be restricted to bounded open convex subsets (or bounded closed convex subsets) of $\mathbb{R}^q$; Hausdorff indicates on p. 160 that if $\ell(U)$ verifies some simple conditions then in his more general situation also the sets U can be restricted to open or closed subsets without changing the value of $L(A)$. The point is that if the sets of the family $\mathcal{U}$ are Borel sets then the construction given in (1) and (2) automatically yields L which satisfies V as well.

The greater part of Carathéodory's fundamental paper [C 1914] is devoted to the study of abstract metric outer measures in $\mathbb{R}^q$; the only concrete example Carathéodory discusses in detail is the case of the linear measure mentioned above and it is shown that this measure gives an appropriate generalisation of the elementary notion of the length of curves. Carathéodory briefly mentions the possibility of p-dimensional measures in $\mathbb{R}^q$ (p being an integer with $1 \leq p \leq q$) by replacing $\ell(U)$ by a "p-dimensional volume" of U as indicated in example (G) of HAUSDORFF (p. 162), the idea being that of generalising ordinary p-dimensional volumes of general subsets of $\mathbb{R}^q$. At this point, HAUSDORFF, by restricting himself to $\ell(U)$ of the form $\lambda(d(U))$ where λ is any non-negative *continuous strictly increasing function* of a real variable $x \in [0, \infty[\quad \lambda : [0, \infty[\to [0, \infty[)$ *with* $\lambda(0) = 0$ (such a λ will be called a *Hausdorff function*) constructs a family of regular metric outer measures L^λ (following the formulae (1) and (2)) using $\mathcal{U}$ as the family of balls (say open balls to fix ideas). HAUSDORFF then shows that if $\lambda(x) = x^p$ and $p = 1$ or 2 then the corresponding L^λ-measure reduces to the CARATHÉODORY linear measure or the usual 2-dimensional surface measure (for sets A in $\mathbb{R}^3$ which are smooth surface elements); further, if $p = q$ then, except for a constant factor, L^λ is the usual LEBESGUE outer measure in $\mathbb{R}^q$. Indeed for any $p = 1, 2, \ldots$ HAUSDORFF's L^λ-measure with $\lambda(x) = x^p$ leads to CARATHÉODORY's p-dimensional measure in $\mathbb{R}^q$, except for constant factors; the proofs of these facts are only briefly outlined. These facts are used essentially to justify the study of his L^λ-measures; we shall call L^λ the *Hausdorff measure* associated with the HAUSDORFF function λ; if $\lambda(x) = x^p, p > 0$, then we shall write $L^{(p)}$ for the corresponding HAUSDORFF measure L^λ.

Now HAUSDORFF introduces his notion of dimension; given a HAUSDORFF function λ, a set $A \subset \mathbb{R}^q$ is said to be of *dimension* $[\lambda]$ if

$$0 < L^\lambda(A) < \infty. \tag{4}$$

As HAUSDORFF points out immediately, the condition (4) depends only on the behaviour of λ in an arbitrarily small neighbourhood of $x = 0$; if λ, μ are two HAUSDORFF functions with

$$0 < a \leq \frac{\mu(x)}{\lambda(x)} \leq b < \infty$$

for x in some interval $]0, \varepsilon[$ then a set A is of dimension $[\lambda]$ if and only if it is of dimension $[\mu]$.

HAUSDORFF now considers the following basic problem : *given a Hausdorff function λ does there exist a set A in $\mathbb{R}^q$ of dimension $[\lambda]$?*

HAUSDORFF does not solve this problem in all generality. Let us state first his most important result in this direction: *if* $\lambda : [0, \infty[\to [0, \infty[$ *is a strictly increasing, strictly concave function with* $\lambda(0) = 0$, $\lim_{x\to\infty} \lambda(x) = \infty$ *then there exists a bounded perfect non-dense set* $A \subset \mathbb{R}$ *of dimension* $[\lambda]$.

Before describing HAUSDORFF's construction of A, let us note that what HAUSDORFF calls *konvex (nach oben)* is what we call (following standard modern terminology) concave and what HAUSDORFF calls *konkav* is what we call convex; if λ is twice differentiable then λ is concave (convex) in modern terminology if $\lambda'' \leq 0$ ($\lambda'' \geq 0$). Note further that a concave real function defined in $]0, \infty[$ is automatically continuous; the determinant condition (γ) on p. 167 is known to be equivalent to the strict concavity of λ. Further, a strictly concave function $\lambda : [0,\infty[\to [0,\infty[$ is known to be strictly subadditive as HAUSDORFF proves in (ι), p. 168, and as can be easily seen from the fact that such a λ with $\lambda(0) = 0$ is of the form

$$\lambda(x) = \int_0^x f(t)dt, \quad x \geq 0$$

where f is strictly decreasing and integrable on $[0, x], x > 0$. If λ is also strictly increasing then $f(t) > 0$ if $t > 0$; since f is strictly decreasing, $f(0+) = \lim_{t\downarrow 0} f(t)$ is in $]0,\infty]$; the case where $f(0+) < \infty$ gives λ such that

$$0 < \lim_{x\to 0} \frac{\lambda(x)}{x} = f(0+) < \infty$$

so that, in this case, by a remark made above, a set $A \subset \mathbb{R}$ is of dimension $[\lambda]$ if and only if its LEBESGUE outer measure is finite and positive. This case is obviously of little interest for the problem at hand; the interesting case is where $f(0+) = \infty$ so that

$$\lim_{x\to 0} \frac{\lambda(x)}{x} = \infty. \tag{5}$$

The relevance of these remarks will become clear later.

The construction given by HAUSDORFF of the set $A \subset \mathbb{R}$ of dimension $[\lambda]$ follows the classical method for the construction of the CANTOR set. One fixes a sequence of strictly positive numbers $\xi_0, \xi_1, \xi_2, \ldots$ with

$$\xi_0 > 2\xi_1,\ \ \xi_1 > 2\xi_2, \ldots;\ \ 2^n\lambda(\xi_n) = 1, \quad n \geq 0 \tag{6}$$

which is easily shown to be possible because of the conditions imposed on λ. From the interval $[0, \xi_0]$ a central open interval of length $\xi_0 - 2\xi_1$ is removed leaving behind two closed intervals, each of length ξ_1; the process is continued with each of the two remaining intervals, leaving behind four closed intervals, each of length ξ_2; at the nth stage one would have 2^n closed intervals left, each of length ξ_n; if the union of these nth stage intervals is called A_n then A is defined as

$$A = \bigcap_{n=1}^{\infty} A_n.$$

It is then obvious that A is a bounded perfect non-dense set with $L^\lambda(A) \leq 1$; the difficult part of HAUSDORFF's work is to show that $L^\lambda(A) > 0$; indeed, he shows that $L^\lambda(A) = 1$.

By taking $\lambda(x) = x^p, 0 < p < 1$, HAUSDORFF obtains A such that $L^{(p)}(A) = 1$; if $p = \log 2/\log 3$, then the numbers ξ_n in (6) are easily seen to be

$$\xi_n = \xi^n, \xi = 2^{-1/p} = \frac{1}{3}$$

so that the set A constructed above in this case becomes the classical CANTOR ternary set in $[0,1]$.

As HAUSDORFF points out, for his construction it suffices to have λ with the requisite properties only in some interval $[0,x_0]$ with $x_0>0$; this allows him to affirm that for

$$\lambda(x)=\left(\log\frac{1}{x}\right)^{-p}$$

his construction will give a set A of dimension $[\lambda]$. It is further implied that this construction will provide sets of dimension $[\lambda]$ for all the functions λ of his logarithmic scale

$$\lambda(x)=x^{p_0}\left(\ell\left(\frac{1}{x}\right)\right)^{-p_1}\left(\ell_2\left(\frac{1}{x}\right)\right)^{-p_2}\cdots\left(\ell_k\left(\frac{1}{x}\right)\right)^{-p_k}$$

where $\ell=\log, \ell_2=\log\log$ etc. and either $p_0=0$ and the first non-vanishing $p_i>0$ or $0<p_0<1$ or $p_0=1$ and the first non-vanishing $p_i<0$. Note that in all these cases λ satisfies (5) and that $\lambda'(x)\to\infty$ as $x\to 0+$ and $\lambda''(x)<0$ for all sufficiently small $x>0$. However, the question of exactly which functions λ will permit sets $A\subset\mathbb{R}$ of dimension $[\lambda]$ is left open. This problem is settled completely by DVORETZKY in 1946; in [D 1948] it is proved that *the necessary and sufficient condition that there exist a set $A\subset\mathbb{R}$ of dimension $[\lambda]$ is that*

$$\liminf_{x\to 0+}\frac{\lambda(x)}{x}>0. \tag{7}$$

As DVORETSKY points out, if the lim inf in (7) is zero then $L^\lambda(S)=0$ for any bounded set S in $\mathbb{R}$ and if the lim inf in (7) is a finite positive number α, then for any bounded interval I of length c, $L^\lambda(I)=\alpha c$; thus the significant part of the statement implied by (7) is to establish that if the lim inf in (7) is ∞ (i. e. λ satisfies (5)) then there exists a bounded set $A\subset\mathbb{R}$ with $0<L^\lambda(A)<\infty$; DVORETZKY's construction is similar to (but more elaborate than) HAUSDORFF's and yields a bounded perfect non-dense set A. DVORETZKY also remarks that *for the existence of $A\subset\mathbb{R}^q$ of dimension $[\lambda]$ it is necessary and sufficient that*

$$\liminf_{x\to 0+}\frac{\lambda(x)}{x^q}>0. \tag{8}$$

It is further observed in [D 1948] that the continuity and even the monotonicity of λ can be dispensed with; actuallly, HAUSDORFF in his paper had already mentioned that the continuity and monotonicity of λ were just convenient hypotheses, implying that it was the rate of convergence to zero of $\lambda(x)$ as $x\to 0+$ which was essential for dimension $[\lambda]$. In [R 1998] p. 68, ROGERS gives a definitive answer to the question of the existence of a set A of dimension $[\lambda]$; *for any Hausdorff function λ there is a compact metric space (X,ρ) such that X is of dimension $[\lambda]$*. In ROGER's proof, X is realized as a compact subset of $[0,1]$ in $\mathbb{R}$ but it is endowed with a suitable new metric ρ; an interesting feature of this construction is that the topology given by ρ in X is identical with that induced by the usual topology of $\mathbb{R}$. This illustrates in a vivid manner the great dependence of the notion of dimension $[\lambda]$ on the choice of the metric ρ, a point which might have interested HAUSDORFF. Indeed if

$$\lambda(x) = \exp(-1/x), \quad x > 0, \quad (\lambda(0) = 0) \tag{9}$$

a function explicitly mentioned by HAUSDORFF, p. 166,

$$\lim_{x \to 0+} \frac{\lambda(x)}{x^q} = 0$$

for any $q \in \mathbb{R}$ so that for any bounded set A in any $\mathbb{R}^q$ (endowed with its Euclidean metric) we should have $L^\lambda(A) = 0$. HAUSDORFF interprets this as saying that dim$[\lambda]$ "corresponds to an infinitely big dimension" (*"entspricht eine unendlich grosse Dimension")*. We should add that in [R 1998], the HAUSDORFF function λ are supposed to be monotonic increasing and only continuous on the right with $\lambda(0) = 0$; an inspection of the proof in [R 1998] seems to imply that no condition on λ beyond $\lambda(x) \to 0$ as $x \to 0+$ is needed to establish ROGER's theorem.

As regards the construction of sets A in $\mathbb{R}^2$ (endowed with the usual Euclidian metric) of dimension $[\lambda]$, λ being a prescribed HAUSDORFF function, HAUSDORFF states (p. 177) that it is *"nicht ohne Schwierigkeit"*. However, he shows (rather easily) that if A is a set of dimension $[\lambda]$ and B is a set of dimension $[\mu]$, A, B *being the special type of sets in* $\mathbb{R}$ *constructed by him previously* then $A \times B$ will be of dimension $[\lambda\mu]$. Here HAUSDORFF comes across a problem which is difficult and which would tax the ingenuity of many mathematicians later: if A is of dimension $[\lambda]$ and B is of dimension $[\mu]$ what can be said of the dimension of $A \times B$? More information on this subtle question is indicated in [R 1998], p. 131.

Let us note that HAUSDORFF's notion of dimension is one of prescribing a suitable HAUSDORFF function λ to the set concerned; what is commonly called *Hausdorff* (fractional) *dimension* these days is a less refined concept. If A is a subset of a metric space (X, ρ), the HAUSDORFF dimension of A is a quantity $\alpha, 0 \leq \alpha \leq \infty$, defined as follows:

$$\alpha = \sup\{p > 0 : L^{(p)}(A) = \infty\} = \inf\{p > 0 : L^{(p)}(A) = 0\}. \tag{10}$$

In other words, the definition (10) is based only on the HAUSDORFF measures defined by the special HAUSDORFF functions $\lambda(x) = x^p, \ p > 0$; that (10) gives a well-defined number α is seen from the easily proved fact that if $L^{(p)}(A) = \infty$ then $L^{(p\prime)}(A) = \infty$ for all p' with $0 < p' < p$ and if $L^{(p)}(A) = 0$ then $L^{(p')}(A) = 0$ for all $p' > p$. Further, if for some $p, \ 0 < p < \infty$, a set A is such that

$$0 < L^{(p)}(A) < \infty$$

then $L^{(p')}(A) = 0$ for $p' > p$ and $L^{(p')})(A) = \infty$ if $0 < p' < p$ so that the HAUSDORFF dimension of A (in the sense of (10)) is equal to p; however, a set

A may be of HAUSDORFF dimension $p,\ 0 < p < \infty$, and yet $L^{(p)}(A)$ may be or 0 or ∞. Thus saying that the classical CANTOR ternary set C is of HAUSDORFF dimension $\log 2/\log 3$ is less precise than the fact proved by HAUSDORFF (see above) that $L^{(p)}(C)$ is finite and > 0 (indeed equal to 1) if $p = \log 2/\log 3$. Nevertheless, for many problems, a knowledge of α (as defined by (10)) is a sufficiently refined piece of information. On the other hand it must be added that there are sets A for which there is no HAUSDORFF function λ such that A is of dimension $[\lambda]$; however, for A arising in many concrete problems, a suitable HAUSDORFF function λ such that A is of dimension $[\lambda]$ exists and then one has indeed the most refined metrical information about the size of A that one would wish to possess.

The present paper of HAUSDORFF seems to have arisen from a close scrutiny of CARATHÉODORY's paper [C 1914]; he did not seem to have any immediate application elsewhere in view. However, it must be added that HAUSDORFF was always interested in the topological notion of dimension (cf. his reference to FRÉCHET's work on p. 157), a subject that had begun to come into its own through the important papers of BROUWER (1913) and LEBESGUE (1911) (cf. [HW 1941], chapter 1). We shall briefly report at the end on the huge amount of unpublished material that HAUSDORFF has left behind on topological dimension theory. Let us only recall at this point that HAUSDORFF was also interested in working out rigorously the notions of length, area, volume etc. as used in all areas of mathematics; this is clearly seen in the introduction to the tenth chapter of his *Grundzüge* ([H 1914a], *Inhalte von Punktmengen*). His presentation of the relevant measure theory there was based on LEBESGUE's 1902 thesis and LEBESGUE's 1904 book (*Leçons sur l'intégration*) and was somewhat laborious; CARATHÉODORY's paper [C 1914] must have seemed to him then (as it still seems to many of us today) an enormous clarification of the theory.

Although HAUSDORFF never published anything more on his own dimension theory and his publications after 1919 turned to matters in entirely different fields of analysis, the after-effects of this 1919 paper were enormous. It would be impossible, if not foolish, to try to give explicit individual references to all the papers which stemmed directly or indirectly from the present paper of HAUSDORFF; we shall therefore limit ourselves to those references which themselves contain substantial bibliographies related to HAUSDORFF measures.

An early explicit reference to HAUSDORFF's present paper is in a paper of BOULIGAND on the DIRICHLET problem in 1925; for this see [CF 1993] which contains English translations of a number of early papers related to fractals, including a translation of the present paper of HAUSDORFF. BOULIGAND published several other papers in 1928 and 1929 on his notion of dimension in relation to his work on potential theory and he seemed to have come to the conclusion that HAUSDORFF's dimension was not related to his definitions. However, in 1935, FROSTMAN in his famous thesis on potential theory [Fr 1935] established a close relationship between the notion of capacity (of a closed set in an Euclidean space) and the HAUSDORFF measure L^{λ} (of the closed set for a suitable HAUSDORFF function λ, cf. [Fr 1935], p. 86); further, both of these were closely related to the notion of transfinite diameter of sets (in $\mathbb{R}^2$, due to FEKETE in 1923, in $\mathbb{R}^3$ due to PÓLYA and SZEGÖ in 1931; [Fr 1935], p. 44–46). By introducing HAUSDORFF's fractional dimension as indicated above in (10), FROSTMAN establishes the equality between the fractional dimension and the capacitory dimension for every closed set in $\mathbb{R}^q$ ([Fr 1935], p. 90); this equality has become a powerful analytical method for calculating the HAUSDORFF fractional dimension of sets in $\mathbb{R}^q$. FROSTMAN's paper shows a thorough study of HAUSDORFF's present paper.

A spate of profound papers on HAUSDORFF measures began to be written by BESICOVITCH starting around 1927; a partial (and yet a long list) of these papers is given in the very readable textbook of ROGERS [R 1998] which we have already cited above. BESICOVITCH and his numerous excellent collaborators wrote (and continue to write; BESICOVITCH died in 1970) on almost all aspects of HAUSDORFF measure theory, with much emphasis on geometric measure theory and metric aspects of number theory; six of these papers are reproduced in [CF 1993] and they may give some idea of the wide range of topics handled by these authors. Incidentally, the definition of HAUSDORFF fractional dimension (as in (10) above) seems to have appeared first in a 1929 paper of BESICOVITCH.

We shall not attempt to describe the numerous applications of HAUSDORFF measures $L^{(p)}$ (specially with p an integer) in the detailed analysis of functions and measures in $\mathbb{R}^n$ which constitute geometric measure theory. Any reasonably general development of the change of variables formula in multiple integrals and the GAUSS-GREEN-STOKES formula in $\mathbb{R}^n$ inevitably leads to the measures $L^{(p)}$ and the associated notions of rectifiability. A definitive account of this theory is given in FEDERER's monograph [F 1969]; a more recent account of some of this theory is [M 1995]. Let us note that in FEDERER's book, important use is made of at least seven different types of HAUSDORFF measures formed by using various geometrically important quantities $\ell(U)$ in the construction indicated in (1) and (2).

As regards applications to number theory, let us cite the following 1934 result of BESICOVITCH ([CF 1993], p. 147): let E be the set of real numbers x in $]0,1[$ such that

$$\limsup_{n\to\infty} \frac{P(x,n)}{n} \leq p$$

where $0 < p < \frac{1}{2}$ and $P(x,n)$ is the sum of the first n digits in the dyadic expansion of x; then the HAUSDORFF fractional dimension (in the sense of (10)) of E is given by

$$\alpha = \frac{I(p)}{\log 2}, \quad I(p) = -p\log p - q\log q, \quad q = 1-p. \tag{11}$$

BESICOVITCH does not state the value of α in this form; however, the advantage of the form (11) lies in the interpretation of $I(p)$ as the SHANNON information associated with the two point probability space $\{0,1\}$, p being the probability given to 1, $q = 1 - p$ to 0. Note that $I(1/2) = \log 2$. Subsequent research has established numerous formulae relating HAUSDORFF fractional dimensions of sets with SHANNON information associated with appropriate (often discrete) probability spaces and this has led to applications in ergodic theory and coding; an useful short bibliography is contained in FALCONER's foreword in [R 1998]. Note that by the BOREL-HAUSDORFF law of normal numbers, the LEBESGUE measure of the set E (for any value of $p < \frac{1}{2}$) is zero; the HAUSDORFF fractional dimension measures the size of E as a function of p. This is a typical service rendered by HAUSDORFF dimensions in general and they are used in this sense in many refined investigations; for example, many curves that arise in various studies of dynamical systems or stochastic processes have no tangents anywhere and their HAUSDORFF dimension functions indicate the degree of their *fractal* nature. A survey of many of these results and others are contained in the various articles on the subject in [FH 1996].

The perfect CANTOR type set A constructed by HAUSDORFF via the numbers $\{\xi_n\}$ defined by (6) has played an important part in harmonic analysis. In the 1940's SALEM used A (and its generalisations) to construct singular probability measures μ supported by A whose FOURIER-STIELTJES coefficients tend to zero; in fact μ is nothing but HAUSDORFF's measure L^λ. Since L^λ has a very simple probabilistic interpretation, its FOURIER-STIELTJES transformation can be written down almost immediately; if $\xi_0 = 1, L^\lambda$ is the probability law of the random variable

$$\eta = \sum_{k=1}^{\infty} \varepsilon_k r_k, \quad r_1 = (1 - \xi_1), \quad r_k = (1 - \xi_k)\xi_1 \ldots \xi_{k-1} \quad (k \geq 2)$$

where ε_k's form a sequence of independent, identically distributed random variables taking the values 0 or 1 with probability 1/2. Thus the FOURIER-STIELTJES transform of L^λ is given by

$$\int_0^1 \exp(it\,x)dL^\lambda(x) = \exp(it/2) \prod_{k=1}^{\infty} \cos(r_k t/2)$$

(using the fact that $\sum_{k \geq 1} r_k = 1$).

The elegant book by KAHANE and SALEM [KS 1994] uses sets A and their generalisations to answer a large number of questions of classical FOURIER analysis; it gives a good introduction to HAUSDORFF measures and the capacity formula of FROSTMAN; it also establishes HAUSDORFF's dimension result about A (p. 30); KAHANE and SALEM call the sets A *"ensembles parfaits symétriques"* (p. 13).

Since MANDELBROT introduced the notion of *fractal objects*, more and more areas of physics and other natural sciences are using concepts related to HAUSDORFF measures. A very complete bibliography can be found in MANDELBROT's book [Ma 1983].

To complete our rapid survey of the after-effects of HAUSDORFF's paper, let us mention its influence on topological dimension theory; to our knowledge, it has not been great. Chapter VII of [HW 1941] is devoted to this topic. As pointed out there, both the CANTOR triadic set as well as the set of irrational numbers in [0, 1] have topological dimension zero; however, we have seen that the CANTOR set has HAUSDORFF fractional dimension $\log 2/\log 3$ (which is bigger than 0.63) and the HAUSDORFF fractional dimension of the irrational numbers in [0, 1] (or in $\mathbb{R}$) is obviously 1 since its 1-dimensional (LEBESGUE) measure is > 0. The basic theorem establishing a relationship between the two notions of dimension was established in 1937 by the Polish analyst SZPILRAJN (who published under the name of EDWARD MARCZEWSKI, after 1945). We state one of its consequences as follows: let $\dim X$ denote the topological dimension of the separable metric space (X, ρ) (defined inductively, $\dim X$ is always an integer ≥ 0 or ∞ and is a topological invariant; it does not depend on the choice of the particular metric ρ); let $\alpha(X, \rho')$ be the HAUSDORFF fractional dimension of (X, ρ') where ρ' is any metric on X such that (X, ρ') is homeomorphic to (X, ρ) as topological spaces; then

$$\dim X = \inf_{\rho'} \alpha(X, \rho') \tag{12}$$

where the infimum is taken over all possible metrics ρ' in X verifying the condition prescribed.

From HAUSDORFF's unpublished papers we know that he remained interested in topological dimension theory all through his life. There are several hundred pages of studies on the ongoing work on dimension theory of HUREWICZ, MENGER, P. ALEXANDROFF, URYSOHN, KURATOWSKI and others. NL HAUSDORFF: Kapsel 47: Fasz. 986 (written between 1930–1936) is a book length study of approx. 200 pages incorporating many of the results of the mathematicians mentioned. All of this shows his permanent interest in dimension theory as a point-set topologist; but we have not found any further work on his own measure theoretical dimension theory.

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